

# GROUP ALGEBRAS WHOSE GROUPS OF NORMALIZED UNITS HAVE EXPONENT 4

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*Abstract.* We give a full description of locally finite 2-groups  $G$  such that the normalized group of units of the group algebra  $FG$  over a field  $F$  of characteristic 2 has exponent 4.

*Keywords:* group of exponent 4, unit group, modular group algebra

*MSC 2010:* 16S34, 16U60

## 1. INTRODUCTION AND RESULT

It is well known that there does not exist an effective description of finite groups of prime square exponent  $p^2$  (not even in the case when the exponent is 4). However Z. Janko (see for example [9, 10, 11]) was able to characterize these groups under certain additional restrictions on their structure. In this way he obtained interesting classes of finite  $p$ -groups.

Note also that there is no effective description of finite 2-groups with pairwise commuting involutions. On the other hand, the structure of a locally finite 2-group  $G$  is known when its normalized group of units  $V(FG)$  of the group algebra  $FG$  has the property that its involutory units pairwise commute (see [5]).

There is a similar situation in the case of powerful  $p$ -groups. Despite of extensive current research of this field, their structure has been incompletely described. However, it is possible to determine [4] those cases when the normalized groups of units of the group algebras are powerful  $p$ -groups.

So it is a natural question whether it is possible to give a description of those modular group algebras whose groups of normalized units have exponent  $p^2$ . In this note we deal with the case of  $p = 2$ .

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Supported by UAEU grants: UPAR G00001922 and StartUP G00001889

Note that in the case when  $p$  is even, 2-groups of exponent 4 appear in several problems in group theory and in the theory of group ring units (see for example [2, 6, 7, 12, 13]).

Our main result is the following.

**Theorem.** *Let  $V(FG)$  be the normalized group of units of a group algebra  $FG$  of a locally finite nonabelian 2-group  $G$  over a field  $F$  with  $\text{char}(F) = 2$ . The group  $V(FG)$  has exponent 4 if and only if  $G = H \times W$ , where  $H$  is a nilpotent group of class 2, the Frattini subgroup of  $H$  is central elementary abelian,  $|H'| \leq 4$ , and  $W$  is an abelian group of exponent at most 4.*

## 2. PRELIMINARIES AND THE PROOF OF THE THEOREM

An involution in a group  $G$  is an element of order 2. For any  $a, b \in G$ , we denote  $(a, b) = a^{-1}b^{-1}ab$  and  $a^b = b^{-1}ab$ . Let  $D_8$  and  $Q_8$  be the dihedral and quaternion groups of order 8, respectively. Define the following groups:

$$\begin{aligned} G_{16}^3 &= \langle g, h \mid g^4 = h^2 = 1, (g^2, h) = 1, \\ &\quad (gh)^3 = hg^3 \rangle \cong (C_4 \times C_2) \rtimes C_2; \\ G_{16}^4 &= \langle g, h \mid g^4 = h^4 = 1, g^h = g^3 \rangle \cong C_4 \rtimes C_4; \\ G_{32}^2 &= \langle g, h \mid g^4 = h^4 = (gh)^2 = 1, \\ &\quad (g^2, h) = (g, h^2) = 1 \rangle \cong (C_4 \times C_2) \rtimes C_4; \\ G_{32}^6 &= \langle g, h \mid g^4 = h^4 = (g^3h)^2 = 1, \\ &\quad (g^2, h) = (g, h^2) = 1 \rangle \cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_2. \end{aligned}$$

For the designation of these groups  $G_m^s$  we use their numbers  $s$  in the Small Groups Library of order  $m$  in the computer algebra program GAP [1].

We use freely the following well known equations (see [8], p. 171)

$$(2.1) \quad (a, bc) = (a, b)(a, c)(a, b, c), \quad (ab, c) = (a, c)(a, c, b)(b, c).$$

**Lemma 1.** *Let  $\text{char}(F) = 2$  and let  $H$  be a nonabelian two-generated subgroup of a group  $G$ . If  $V(FG)$  has exponent 4, then  $H \in \{D_8, Q_8, G_{16}^3, G_{16}^4, G_{32}^2\}$ . Moreover  $H' \subseteq \Phi(H) \subseteq \zeta(H)$ .*

*Proof.* Clearly  $G$  has exponent 4 and any two involutions in  $G$  either commute or generate a dihedral group  $D_8$  of order 8. Put  $H = \langle g, h \mid g, h \in G, (g, h) \neq 1 \rangle$ . Consider the following cases:

Case A. Let  $H = \langle g, h \mid |g| = 4, |h| = 2 \rangle$ , such that  $H \not\cong D_8$ . Then  $x = 1 + g + h \in V(FG)$  has order 4 and

$$\begin{aligned} x^4 - 1 &= (gh)^2 + (hg)^2 + g^3h + ghg^2 \\ &\quad + g^2hg + \underline{hg^3} + g^2 + hg^2h = 0. \end{aligned}$$

Comparing  $hg^3$  with other elements, from the last equation we get  $hg^3 = g^2hg$ , so

$$(2.2) \quad (h, g^2) = 1 \quad \text{and} \quad (gh)^2 = (hg)^2.$$

It is easy to see  $G/\langle g^2 \rangle$  is generated by two involutions  $g\langle g^2 \rangle$  and  $h\langle g^2 \rangle$ . In the case when  $G/\langle g^2 \rangle$  is abelian we have  $(g, h) = g^2$ , it follows that  $hgh = g^3$  and  $\langle g, h \rangle \cong D_8$ , a contradiction. Therefore,  $G/\langle g^2 \rangle \cong D_8$ . Since  $G$  is not  $D_8$ , by (2.2) we obtain that  $(gh)^2 = (hg)^2 \neq 1$ . Consequently,  $(gh)^3 = hg^3 = (gh)^{-1}$  and

$$H = \langle g, h \mid g^4 = h^2 = 1, (g^2, h) = 1, (gh)^4 = 1 \rangle \cong G_{16}^3.$$

Case B. Let  $H = \langle g, h \mid |g| = |h| = 4, |gh| = 2 \rangle$ . Clearly the unit  $y = 1 + g + gh \in V(FG)$  has order 4. Since  $ghg = h^3$ , we have that  $y^2 = g^2 + g^2h + h^3$  and

$$\begin{aligned} y^4 - 1 &= gh^3gh + h^2 + \underline{h} + gh^3g \\ &\quad + g^2h^3 + h^3g^2 + g^2 + h^3g^2h = 0. \end{aligned}$$

Comparing  $h$  with other elements, from the last equation we obtain that only  $h = gh^3g$ , so  $(h, g^2) = 1$ . Consequently

$$H = \langle g, h \mid g^4 = h^4 = (gh)^2 = 1, (g^2, h) = 1 \rangle \cong G_{16}^3.$$

Case C. Let  $H = \langle g, h \mid |g| = |h| = |gh| = 4 \rangle$  such that  $H \not\cong Q_8$ . Then  $x = 1 + g + h \in V(FG)$  has order 4 and

$$\begin{aligned} (2.3) \quad x^4 - 1 &= (gh)^2 + (hg)^2 + \underline{g^3h} + ghg^2 + g^2h^2 \\ &\quad + h^2g^2 + g^2hg + hg^3 + h^2gh + gh^3 \\ &\quad + h^3g + hgh^2 + gh^2g + hg^2h = 0. \end{aligned}$$

The element  $g^3h$  must coincide with one of the following elements:

Case 1. Let  $g^3h = (hg)^2$ . Clearly,  $h = g(hg)^2$  and  $h^2 = (gh)^3$ , so  $|gh| = 2$ , a contradiction.

Case 2. Let  $g^3h = ghg^2$ . Then  $(h, g^2) = 1$  and (2.3) can be rewrite as

$$(gh)^2 + (hg)^2 + h^2gh + gh^3 + h^3g + hgh^2 + gh^2g + g^2h^2 = 0.$$

Then  $(g, h^2) = 1$  and  $(gh)^2 = (hg)^2$ . It follows that

$$H = \langle g, h \mid g^4 = h^4 = (g, h^2) = (g^2, h) = 1, (gh)^2 = (hg)^2 \rangle \cong G_{32}^2.$$

Case 3. Let  $g^3h = h^2g^2$ . Then  $h = gh^2g^2$  and  $hg = gh^2g^3$ , so  $(hg)^2 = 1$  which is impossible.

Case 4. Let  $g^3h = h^2gh$  or  $g^3h = gh^3$ . Then  $g^2 = h^2$  and by (2.3)  $(gh)^2 = (hg)^2$ . It follows that

$$H = \langle g, h \mid g^4 = h^4 = 1, g^2 = h^2, (gh)^2 = (hg)^2 \rangle \cong G_{16}^4.$$

Case 5. Let  $g^3h = hgh^2$ . Then  $|gh| = 2$ , a contradiction.

Case 6. Let  $g^3h = gh^2g$ . Then  $h = g^2h^2g$ , so  $gh = g^{-1}(h^2)g$  and  $2 = |h^2| = |gh|$ , a contradiction.

Case 7. Let  $g^3h = h^3g$ . Then  $gh^3 = hg^3$  and by (2.3) we get that

$$(2.4) \quad \begin{aligned} & ghgh + hghg + \underline{ghg^2} + g^2h^2 + h^2g^2 + \\ & + g^2hg + h^2gh + gh^2g + hgh^2 + hg^2h = 0. \end{aligned}$$

It is easy to check that  $ghg^2 \in \{(hg)^2, h^2gh, hgh^2, hg^2h\}$ . We consider each case separately.

Case 7.1. Let  $ghg^2 = hghg$ . Then from (2.4) follows that

$$(2.5) \quad g^2h^2 + h^2g^2 + hgh^2 + \underline{h^2gh} + gh^2g + hg^2h = 0.$$

It is easy to check that only possible cases are  $h^2gh \in \{g^2h^2, gh^2g\}$ .

If  $h^2gh = g^2h^2$  then  $g^2h = h^2g$  and  $h^3g = g^3h = g(g^2h) = g(h^2g)$ , so  $h = g$ , a contradiction.

If  $h^2gh = gh^2g$  then  $h(hgh) = (ghg)g^3hg$ , then  $h(ghg) = (hgh)g^3hg$ , so  $1 = ghg^2$ , a contradiction.

Case 7.2. Let  $ghg^2 = h^2gh$ . Multiplying it on the left side by  $g^2$  and on the right side by  $gh$  we obtain that  $1 = (g^3h)^2 = g^2h^2ghgh$ . Since  $|gh| = 4$ , this yields that  $g^2h^2 = ghgh$  and  $(g, h) = 1$ , a contradiction.

Case 7.3. Let  $ghg^2 = hg^2h$ . Multiplying it on the left side by  $g^2$  and on the right side by  $gh$  we obtain that  $1 = (g^3h)^2 = g^2hg^2hgh$ . Since  $|gh| = 4$ , this yields that  $g^2hg = ghgh$  and  $ghg = hgh$ . Clearly  $hghg = ghg^2 = hg^2h$ , so  $(g, h) = 1$ , a contradiction.

Case 7.4. Let  $ghg^2 = hgh^2$ . Then

$$\begin{aligned} H &= \langle g, h \mid g^4 = h^4 = 1, (g^3h)^2 = 1, ghg^2 = hgh^2 \rangle \\ &\cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_2 \cong G_{32}^6. \end{aligned}$$

Put  $w = 1 + g(1 + h) \in V(FH)$ . It is easy to check that

$$w^2 = 1 + g^2 + (gh)^2 + g^2h + ghg.$$

The powers of  $w$  and their orders are easy to calculate using the package LAGUNA of the computational algebra system GAP [1].

However we assume  $w^4 = 1$ . By a straightforward calculation

$$\begin{aligned} w^4 &= h + g^2 + (gh)^4 + (g^3h)g + g(ghg^2) + (ghg^2)hg \\ &\quad + g^2(hg)^2 + g(hg^3) + (g^2h)^2 \\ &= h + g^2 + (gh)^4 + h^3g^2 + g(hgh^2) + (hgh^2)hg \\ &\quad + g^2(hg)^2 + g^2h^3 + (g^2h)^2 = 1. \end{aligned}$$

Comparing the element  $h$  with other elements we have  $h = g^2(hg)^2$ . This yields that  $hg = g^2h(ghg^2) = g^2h(hgh^2)$  and  $g^{-1}hg = (gh^2)^2$ . However  $4 = |h| > |(gh^2)^2| \leq 2$  because  $\exp(G) = 4$ , a contradiction. Consequently  $\exp(V(FH)) > 4$ , which is impossible.  $\square$

**Corollary 1.** *If  $\exp(V(FG)) = 4$ , then  $G' \leq \Phi(G) \leq \zeta(G)$ ,  $\Phi(G)$  is elementary abelian and  $G$  has nilpotency class 2.*

*Proof.* Let  $H = \langle a, b \in G \mid c = (a, b) \neq 1 \rangle$ . Clearly  $c = g_1^2 g_2^2 \cdots g_n^2$  for some  $g_1, \dots, g_n \in G$  (see Theorem 10.4.3 in [8], p.178).

Using induction on  $n \geq 1$ , let us prove that  $(c, x) = (g_1^2 \cdots g_n^2, x) = 1$  for any  $x \in G$ .

Base of induction:  $n = 1$ . Then  $(g_1^2, x) = (g_1, x)(g_1, x, g_1)(g_1, x) = (g_1, x)^2 = 1$  by (2.1) and Lemma 1.

Put  $w = g_1^2 g_2^2 \cdots g_{n-1}^2$ . Using the same arguments, (2.1) and Lemma 1

$$(wg_n^2, x) = (w, x)(w, x, g_n^2)(g_n^2, x) = (g_n^2, x) = (g_n, x)^2 = 1.$$

$\square$

**Lemma 2.** *Let  $G$  be a finite 2-group, such that its Frattini subgroup  $\Phi(G)$  is central elementary abelian,  $G' \leq \Phi(G)$  and  $|G'| \leq 4$ . If  $\text{char}(F) = 2$  then the exponent of  $V(FG)$  is equal 4.*

*Proof.* Let  $G = g_1\Phi(G) \cup \cdots \cup g_m\Phi(G)$ , where  $g_1 = 1$ . Then any  $u \in V(FG)$  can be written as  $u = \sum_{i=1}^m g_i u_i$ , where  $u_1, \dots, u_m \in F\Phi(G)$ . By Brauer's lemma ([3], Proposition 3.1, p.17)  $u^2 = \sum_{i=1}^m g_i^2 u_i^2 + \sum_{1 \leq i < j} [g_i, g_j] u_i u_j$ , where  $[g_i, g_j] = g_i g_j - g_j g_i \in FG$ . The element  $\sum_{i=1}^m g_i^2 u_i^2$  is a central involution, so

$$u^4 = 1 + \left( \sum_{1 \leq i < j} [g_i, g_j] u_i u_j \right)^2.$$

Since  $[g_i, g_j] = g_i g_j (1 - (g_j, g_i))$  and  $1 - (g_j, g_i)$  is a central nilpotent element of index 2, by Brauer's lemma ([3], Proposition 3.1, p.17) we have that

$$z = \left( \sum_{1 < i < j}^m g_i g_j \cdot (1 - (g_j, g_i)) u_i u_j \right)^2 = \sum_{1 < i < j, 1 < k < l}^m g_i g_j g_k g_l \times \\ \times (1 - (g_k g_l, g_i g_j)) (1 - (g_j, g_i)) (1 - (g_l, g_k)) u_i u_j u_k u_l.$$

Suppose that  $G' \cong C_2 \times C_2$ . It is easy to check that  $1 - (g_j, g_i) \in \omega(FG')$  and  $\omega(FG')^3 = 0$ . Consequently,  $z = 0$  and  $\exp(V(FG)) = 4$ .  $\square$

**Lemma 3.** *Let  $G = H \times \langle a \rangle$  and  $|a|$  divides 4. If  $\exp(V(FH)) = 4$ , then  $\exp(V(FG)) = 4$ , too.*

*Proof.* First assume that  $|a| = 2$ . Any  $x \in V(FG)$  has the form

$$x = \sum_{g \in H} \alpha_g g + a \sum_{h \in H} \beta_h h$$

and  $g^2, h^2 \in \Phi(G)$  are central by Corollary 1. Clearly, the order of the unit  $y = \sum_{g \in H} \alpha_g g + \sum_{h \in H} \beta_h h$  divides 4 and

$$y^2 = \sum_{g^2 \in H} \alpha_g^2 g^2 + \sum_{h^2 \in H} \beta_h^2 h^2 + \sum_{g, h \in H} \alpha_g \beta_h [g, h].$$

The unit  $\sum_{g^2 \in H} \alpha_g^2 g^2 + \sum_{h^2 \in H} \beta_h^2 h^2$  is central and its order divide 2, so

$$y^4 = 1 + \left( \left[ \sum_{g \in H} \alpha_g g, \sum_{h \in H} \beta_h h \right] \right)^2 = 1$$

and  $(\left[ \sum_{g \in H} \alpha_g g, \sum_{h \in H} \beta_h h \right])^2 = 0$ . It follows that  $|x|$  also divides 4.

Finally let  $|a| = 4$  and  $L = H \times \langle a^2 \rangle$ . Then  $\exp(V(FL)) = 4$  and any  $x \in V(FG)$  has the form  $x = \sum_{g \in L} \alpha_g g + a \sum_{h \in L} \beta_h h$ . Repeating the previous argument it is easy to see that  $|x|$  divides 4.  $\square$

**Lemma 4.** *Let  $\text{char}(F) = 2$  and let  $G$  be a finite 2-group, such that  $G'$  is central elementary abelian. If  $|G'| \geq 8$ , then  $\exp(V(FG)) > 4$ .*

*Proof.* If  $x = \sum_{g \in G} \alpha_g g \in V(FG)$ , then  $x^2 = \sum_{g \in G} \alpha_g^2 g^2 + w_2$ , where  $w_2 \in \langle g_i g_j (1 + (g_i, g_j)) \mid i, j \in \mathbb{N} \rangle_F$ . Since  $g^2 \in \Phi(G) \subseteq \zeta(G)$ , we have

$$x^4 = \sum_{g \in G} \alpha_g^4 g^4 + w_2^2 = \sum_{g \in G} \alpha_g^4 + w_2^2.$$

Using the equalities (2.1) and the fact that  $|G'| \geq 8$ , we have that

$$w_2^2 \in \langle g_i g_j g_k g_l (1 + (g_i g_j, g_k g_l)) (1 + (g_i, g_j)) (1 + (g_l, g_k)) \mid i, j, k, l \rangle_F \neq 0$$

so  $x^4 \neq 1$ . □

*Proof of the Theorem.* Follows from Lemmas 1–4. □

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